ANALYSIS OF TWO-PHASE FLOW INSTABILITIES IN PIPE-RISER SYSTEMS

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ABSTRACT
A differential-algebraic system is presented to model unstable two-phase flows in pipe-riser systems. Equations derive from the space integration of an isothermal drift-flux model assuming quasi-equilibrium momentum balance. A linear analysis of this system gives a new stability criterion for gas-liquid flows in pipe-riser systems. This criterion is validated by laboratory experiments. Then, a nonlinear analysis shows that the severe slugging phenomenon is a hydrodynamic instability coming from a supercritical Hopf bifurcation.

Keywords. Two-phase flow, severe slugging, stability, Hopf bifurcation, differential-algebraic equations.

NOMENCLATURE

\begin{align*}
A & \quad \text{pipe cross-section area} \ [m^2] \\
A_G & \quad \text{cross-section area occupied by gas} \ [m^2] \\
A_L & \quad \text{cross-section area occupied by liquid} \ [m^2] \\
C_0 & \quad \text{distribution parameter} \\
D & \quad \text{pipeline internal diameter} \ [m] \\
G & \quad \text{gas phase} \\
g & \quad \text{gravitational acceleration} \ [m.s^{-2}] \\
H & \quad \text{riser height} \ [m] \\
in & \quad \text{riser inlet} \\
L & \quad \text{pipe length} \ [m] \text{ or liquid phase} \\
m_L & \quad \text{liquid accumulation in the riser} \ [kg.m^{-2}] \\
m_G & \quad \text{gas accumulation in the riser} \ [kg.m^{-2}] \\
out & \quad \text{riser outlet} \\
P & \quad \text{pressure} \ [Pa] \\
P_{atm} & \quad \text{atmospheric pressure} \ [Pa] \\
R_G & \quad \text{gas surface fraction}, \ R_G = A_G/A \\
R_L & \quad \text{liquid surface fraction}, \ R_L = A_L/A = 1 - R_G \\
R & \quad \text{ideal gas constant} \ [m^2.s^{-2}.K^{-1}] \\
S & \quad \text{separator} \\
T & \quad \text{temperature} \ [K] \\
t & \quad \text{time} \ [s] \\
u_d & \quad \text{drift velocity} \ [m.s^{-1}] \\
U_L & \quad \text{liquid superficial velocity} ; \ U_L = R_LV_L \ [m.s^{-1}] \\
U_G & \quad \text{gas superficial velocity} ; \ U_G = R_GV_G \ [m.s^{-1}] \\
U_S & \quad \text{mixture superficial velocity} ; \ U_S = U_L + U_G \\
V_L & \quad \text{liquid velocity} \ [m.s^{-1}] \\
V_G & \quad \text{gas velocity} \ [m.s^{-1}] \\
x & \quad \text{axial coordinate of the pipe} \ [m] \\
z & \quad \text{axial coordinate of the riser} \ [m] \\
\rho & \quad \text{density} \ [kg.m^{-3}] \\
\theta_{p} & \quad \text{pipe inclination with respect to horizontal} \\
\Omega & \quad \text{wetted angle} \\
\sigma & \quad \text{gas-liquid surface tension} \ [N.m^{-1}] \\
0 & \quad \text{pipe inlet} \\
1 & \quad \text{pipe outlet}
\end{align*}
INTRODUCTION

Recent estimations predict that over half offshore oil and gas reserves are located in deep water and marginal fields. For such reserves, economic recovery methods are required. Therefore, multiphase flows are transported within pipelines and separated on treatment platforms built in shallow water or processed in onshore facilities. Unfortunately, hydrodynamic instabilities may occur whenever gas and liquid flow in a pipeline, generating serious operating problems.

THE PHYSICAL PROBLEM

In certain conditions, a steady gas-liquid flow in a pipeline (i.e. a two-phase flow with constant gas and liquid mass flow rates) does not exist (Taitel, 1986). For instance, whenever a sub-sea line ends at a vertical pipe (i.e. a riser) connected to a platform separator, the base of the riser may accumulate some liquid and stop the gas motion. The upstream gas is compressed until its pressure is large enough to push the liquid slug, downstream into the separator.

One of these instabilities is known as the severe slugging phenomenon (e.g. (Taitel et al., 1990), (Zakarian and Tran, 1999)) which is a strongly unstable two-phase flow and one of the most severe case that one may face in multiphase production. Several prediction criteria of the severe slugging phenomenon exist in literature. They suggest the existence of critical values of parameters such as the height of the riser, inlet phase flow rates, etc. When one of these parameters is less (or greater) than a certain critical value, flows are stable (respectively unstable) or vice-versa. These critical values are called bifurcation points.

Around these points, flows are nearly steady (cf. Fig. 1, page 8) (Taitel et al., 1990). The liquid column in the riser is stable but quantities such as the pressure or the gas volume fraction at the riser base tend to oscillate significantly. These fluctuations may dissipate and converge to a steady state. Otherwise, these oscillations may propagate indefinitely in a quasi-steady cyclic process.

A DIFFERENTIAL-ALGEBRAIC MODEL

Quasi-steady flows in pipe-riser systems usually represent a transition between stable and severe slug flows. They may occur whenever liquid and gas flow rates are low.

Let us consider a pipe-riser system connected to a separator which is operated at a constant pressure $P_S$ (cf. Fig. 5, page 9). The gas and liquid superficial velocities at the pipe inlet, namely $U_{L0}$ and $U^0_R$, are also assumed to be low and constant (see nomenclature for more details).

Therefore, according to experiments, the flow pattern is stratified in the pipe and bubble or slug in the riser (cf. Fig. 2 and 3) (Barnea and Taitel, 1986). We suggest that the gas surface fraction (or void fraction) is approximately constant along the pipe, for all time. Its value will be given by the geometrical relation:

$$R^\text{pipe}_G \left( U^0_L, \theta_p, D \right) = 1 - \frac{A_L \left( U^0_L, \theta_p, D \right)}{A}$$

$$= 1 - \frac{1}{2\pi} \left[ \Omega \left( U^0_L, \theta_p, D \right) - \sin \Omega \left( U^0_L, \theta_p, D \right) \right]$$

where the wetted angle $\Omega$ (cf. Fig. 4, page 9) is the unique solution of the following equation (Schmidt et al., 1980):

$$U^0_L - \frac{149}{2\pi} \left[ \frac{D}{4} \right]^{2/3} \sin \theta_p \left[ \frac{1}{2} \left( \Omega - \sin \Omega \right)^{5/3} \right] = 0. \quad (2)$$

Remark: when $R^G_G$ decreases to zero, $\Omega$ tends to $2\pi$. We deduce that $U^0_L$ converges to the critical value:

$$(U^0_L)_{\text{crit}} = 149 \left[ \frac{D}{4} \right]^{2/3} \sin \theta_p^{1/2}. \quad (3)$$

Above this value, the flow pattern in the pipe is dispersed or nearly liquid. Therefore, the severe slugging phenomenon cannot exist (Schmidt et al., 1985).

Assuming a constant liquid density, we deduce from the space integration of the liquid mass conservation equation along the pipe that the liquid superficial velocity at the pipe outlet, namely $U^0_L$, is equal to $U^0_L$, for all time. Since the pipe is slightly inclined, its pressure drop is mainly due to friction. But in our problem, phase velocities are small.

Therefore, the pipe pressure, namely $P_{\text{pipe}}$, is approximately the same along the pipe. It will be given by the following relation:

$$P_{\text{pipe}}(t) = P_S + g \left[ m_L(t) + m_G(t) \right] \quad (4)$$

where $m_L(t)$ and $m_G(t)$ are the liquid and gas accumulation in the riser:

$$m_L = \int_0^H \rho_L R_L dz; \quad m_G = \int_0^H \rho_G R_G dz. \quad (5)$$

Since our results will be compared to laboratory experiments operated at standard conditions, the gas may be assumed perfect:

$$\rho_G(P) = \frac{P}{RT}. \quad (6)$$
\( R \) is the ideal gas constant and \( T \) is the constant flow temperature.

Space integration of the gas mass conservation equation along the pipe leads to the differential equation:

\[
\frac{dP_{pipe}}{dt}(t) = \frac{U_G^0 - U_G^1(t)}{LR_{pipe}^G} P_{pipe}(t) \tag{7}
\]

where \( U_G^0(t) \) refers to gas superficial velocity at the pipe outlet.

Assuming a slug or bubble flow pattern in the riser, the gas velocity will be formulated as (Taitel et al., 1990):

\[
V_G = C_0 U_S + u_d \tag{8}
\]

where

\[
C_0 = \begin{cases} 
1.2 & \text{(slug flow)}, \\
1.0 & \text{(bubble flow)}, 
\end{cases} 
\]

\[
u_d = \begin{cases} 
0.35 \frac{gD}{\rho_L} & \text{(slug flow)}, \\
1.53 \frac{g}{\rho_L}^{1/4} & \text{(bubble flow)}. \tag{9}
\end{cases}
\]

We will also assume continuous superficial velocities of gas and liquid at the pipe-riser connection:

\[
U_G^{in}(t) = U_G^1(t) ; U_L^{in}(t) = U_L^1(t). \tag{11}
\]

Superscripts \( \text{in} \) refer to the riser inlet. Then, space integration of the mass conservation equations of each phase along the riser yields:

\[
\frac{dn_l}{dt} = \rho_L \left\{ U_L^0 - \left[ U_S^{out} - R_{G}^{out} \left( C_0 U_S^{out} + u_d \right) \right] \right\}, \tag{12}
\]

\[
\frac{dn_G}{dt} = \frac{1}{RT} P_{pipe} R_{G}^{in} \left[ C_0 \left( U_L^0 + U_L^{out} \right) + u_d \right] - \frac{1}{RT} P_S R_{G}^{out} \left( C_0 U_S^{out} + u_d \right) \tag{13}
\]

where superscripts \( \text{out} \) refer to the riser outlet.

In order to complete this set of equations, two additional relations derived from Eq. (5) close the system:

\[
0 = m_L - \frac{1}{2} H \rho_L \left[ 1 - R_{G}^{out} + 1 - R_{G}^{out} \right], \tag{14}
\]

\[
0 = m_G - \frac{1}{2} \frac{H}{RT} \left[ P_{pipe} R_{G}^{in} + P_S R_{G}^{out} \right]. \tag{15}
\]

These approximations are reasonable as long as void fraction and pressure oscillations in the riser are not too important, or in other words, as long as the liquid column in the riser is stable (Taitel et al., 1990). This is quite true whenever flows are slightly unstable (Zakarian and Tran, 1999).

Finally, we deduce a differential-algebraic system:

\[
\frac{dX}{dt}(t) = F(X(t), Y(t), p), \tag{16}
\]

\[
0 = G(X(t), Y(t), p) \tag{17}
\]

where \( X \) and \( Y \) refer respectively to the differential and the algebraic variables:

\[
X = \left[ m_L, m_G, P_{pipe} \right]^T, \tag{18}
\]

\[
Y = \left[ U_G^{in}, R_G^{in}, R_G^{out}, U_S^{out} \right]^T. \tag{19}
\]

\( p \) refers to the physical parameters of the system:

- Experimental: \( U_L^0, U_G^0, P_S \).
- Geometrical: \( L, H, D, \theta_p \).
- Thermodynamic: \( \rho_L, \sigma, T \).

\( F \) and \( G \) coordinates are:

\[
F_1(X, Y, p) = \rho_L \left\{ U_L^0 - \left[ Y_4 - Y_3 (C_0 Y_4 + u_d) \right] \right\}, \tag{20}
\]

\[
F_2(X, Y, p) = \frac{1}{RT} X_3 Y_2 \left[ C_0 (U_L^0 + Y_1) + u_d \right] - \frac{1}{RT} P_S Y_3 (C_0 Y_4 + u_d), \tag{21}
\]

\[
F_3(X, Y, p) = \frac{U_G^0 - Y_1}{LR_{pipe}^G} X_3, \tag{22}
\]

\[
G_1(X, Y, p) = P_S - X_3 + g(X_1 + X_2), \tag{23}
\]

\[
G_2(X, Y, p) = X_1 - \frac{1}{2} H \rho_L \left[ (1 - Y_2) + (1 - Y_3) \right], \tag{24}
\]

\[
G_3(X, Y, p) = X_2 - \frac{1}{2} H \frac{\rho_L}{RT} \left[ X_3 Y_2 + P_S Y_3 \right], \tag{25}
\]

\[
G_4(X, Y, p) = Y_1 - Y_2 \left[ C_0 (U_L^0 + Y_1) + u_d \right]. \tag{26}
\]

Remark: whether mass flow rates, namely \( q_L^0 \) and \( q_G^0 \), are imposed at the pipe inlet instead of superficial velocities, we only need to replace \( U_G^0 \) by \((RT q_G^0)/P_{pipe}\) and \( U_L^0 \) by \( q_L^0/\rho_L \) in Eq. (20) - Eq. (26).

INDEX REDUCTION

We notice that the right members Eq. (23) - Eq. (26) do not depend on the algebraic variable \( Y_4 \). Therefore, we
cannot apply the Implicit Function Theorem to the algebraic equations Eq. (17) in order to write $Y$ versus $X$.

One says that the system Eq. (16) - Eq. (17) has an index greater than one (Brenan et al., 1996). Since most of the results in bifurcation theory only concerned systems of index one (i.e. differential-algebraic systems equivalent to ordinary differential systems), our index must be reduced.

Thus, let us take the derivative of the first algebraic equation with respect to $t$:

$$\frac{dX_3}{dt} = g \left[ \frac{dX_1}{dt} + \frac{dX_2}{dt} \right]$$

which is equivalent to:

$$F_3(X, Y, p) = g [F_1(X, Y, p) + F_2(X, Y, p)].$$

Given the expressions of $F_1$, $F_2$ and $F_3$, we deduce $Y_4$ as a function of $X_3$, $Y_1$, $Y_2$ and $Y_3$:

$$Y_4 = \frac{\frac{1}{g} U_0^g - Y_3 - \rho_L(U_0^g + Y_3 u_d)}{\rho_L(-1 + Y_3 C_0) - \frac{P_S}{RT} Y_3 C_0} \frac{\rho_L(-1 + Y_3 C_0) - \frac{P_S}{RT} Y_3 C_0}{\rho_L RT(-1 + Y_3 C_0) - \frac{P_S}{RT} Y_3 C_0} \frac{- X_3 Y_2 \left[ C_0 (U_0^g + Y_1) + u_d \right] - \frac{P_S}{RT} Y_3 u_d}{\rho_L RT(-1 + Y_3 C_0) - \frac{P_S}{RT} Y_3 C_0}.$$  \hspace{1cm} (27)

Therefore, we only need to know $X_1$, $X_2$, $Y_1$, $Y_2$ and $Y_3$ to deduce $X_3$ from $G_1(X, Y, p) = 0$ and $Y_4$ from Eq. (27). Removing the last differential equation and the first algebraic equation from Eq. (16) - Eq. (17), we find a new system:

$$\frac{dx}{dt}(t) = f(x(t), y(t), p), \hspace{1cm} (28)$$

$$0 = g(x(t), y(t), p) \hspace{1cm} (29)$$

where $f$ and $g$ coordinates are:

$$f_1(x, y, p) = \rho_L \left\{ U_0^g - [Y_4(x_1, x_2, y_1, y_2, y_3)] \right\},$$

$$f_2(x, y, p) = \frac{1}{RT} X_3(x_1, x_2) Y_2 \left[ C_0 (U_0^g + Y_1) + u_d \right] - \frac{1}{RT} P_S y_3 \left[ C_0 Y_4(x_1, x_2, y_1, y_2, y_3) + u_d \right], \hspace{1cm} (30)$$

$$g_1(x, y, p) = x_1 - \frac{1}{2} H \rho_L \left[ \left( 1 - y_2 \right) \left( 1 - y_3 \right) \right], \hspace{1cm} (32)$$

$$g_2(x, y, p) = x_2 - \frac{1}{2} \frac{H}{RT} \left[ X_3(x_1, x_2) Y_2 + P_S y_3 \right], \hspace{1cm} (33)$$

$$g_3(x, y, p) = y_1 - y_2 \left[ C_0 (U_0^g + Y_1) + u_d \right]. \hspace{1cm} (34)$$

$x$ and $y$ refer to the new differential and algebraic variables:

$$x = [m_L, m_G]^T, \hspace{1cm} \hspace{1cm} (35)$$

$$y = [U^G, R^G, R^{out}]^T. \hspace{1cm} (36)$$

Contrary to the previous system, Eq. (28) - Eq. (29) has an index equal to one almost everywhere since:

$$\text{Det} \left[ D_y g(x, y, p) \right] = -\frac{1}{4} \left( 1 - y_2 C_0 \right) \frac{H^2 \rho_L}{RT} \left[ P_S - X_3(x_1, x_2) \right].$$

Det $[D_y g(x, y, p)]$ is zero when $y_2 = 1/C_0$, which is impossible for a steady flow since:

$$y_2^{steady} = \frac{U_0^G}{C_0 (U_0^M + U_0^G) + u_d}. \hspace{1cm} (37)$$

According to Eq. (9), $1/C_0 > 0.833$. Therefore, the system Eq. (28) - Eq. (29) is singular whenever $R^{out}_G = 1/C_0 > 0.833$ or in other words, whenever a large bubble penetrates into the riser. According to our assumption of a bubble or slug flow pattern in the riser, $R^{out}_G$ should not exceed 0.76 (Barnea and Brauner, 1985). So we will assume that the system Eq. (28) - Eq. (29) is always equivalent (or transferable (Griepentrog and März, 1986)) to an ordinary differential system whenever the flow is steady or slightly unstable.

**LINEAR ANALYSIS**

Given all the parameters $p$, three equilibrium points are solution of the following system:

$$f(x, y, p) = 0,$$

$$g(x, y, p) = 0.$$
where

\[ P(p) = D_x f(x_0(p), y_0(p), p), Q(p) = D_y f(x_0(p), y_0(p), p), \]
\[ R(p) = D_x g(x_0(p), y_0(p), p), S(p) = D_y g(x_0(p), y_0(p), p). \]

We also define the set:

\[ \sigma(p) = \{ \lambda(p) \in \mathbb{C} | Pol(\lambda, p) = 0 \}. \]

\( \sigma(p) \) refers to the eigenvalues of the system Eq. (28) - Eq. (29) with respect to the equilibrium point \((x_0(p), y_0(p), p)\). One may show that:

\[ \sigma(p) = \{ \alpha(p) + i\omega(p), \alpha(p) - i\omega(p) \} \]

when flows are nearly or slightly unstable; \( \alpha \) and \( \omega \) are constant quantities, depending on \( p \). A classical theorem states that \((x_0(p), y_0(p), p)\) is asymptotically stable if \( \sigma(p) \) is included in the half-plane \( \{ z \in \mathbb{C} : Re(z) < 0 \} \) (Griepentrog and März, 1986). Therefore, instabilities occur whenever for a given set of parameters, \( \alpha(p) \) becomes positive. We deduce the expression of the stability boundary of two-phase flows in pipe-riser systems from the equation \( \alpha(p) = 0 \):

\[ \text{Trace} \left[ P(p) - Q(p) [S(p)]^{-1} R(p) \right] = 0. \quad (39) \]

**NONLINEAR ANALYSIS**

In a pipe-riser system, a slightly unstable flow oscillates periodically. In order to predict its period and the maximal amplitude of its oscillations, a nonlinear analysis is required.

First, let us write the algebraic variables with respect to the differential variables (see Eq. (35) - Eq. (36)). We solve the algebraic equations Eq. (32) - Eq. (34) with respect to \( x_1, x_2 \) and the function \( X_1 \):

\[
y_1(x_1, x_2) = \frac{2C_0 U^0_L}{H \rho^0_L} \left[ x_2 RT \rho^0_L + P_S \left( x_1 - H \rho^0_L \right) \right]
\]
\[ + \frac{2u_d}{H \rho^0_L} \left[ x_2 RT \rho^0_L + P_S \left( x_1 - H \rho^0_L \right) \right], \]
\[
y_2(x_1, x_2) = 2 \frac{P_S \left( x_1 - H \rho^0_L \right) + x_2 RT \rho^0_L}{H \rho^0_L \left[ X_3(x_1, x_2) - P_S \right]}, \]
\[
y_3(x_1, x_2) = -2 \frac{X_3(x_1, x_2) \left( x_1 - H \rho^0_L \right) + x_2 RT \rho^0_L}{H \rho^0_L \left[ X_3(x_1, x_2) - P_S \right]}, \]

Replacing \( y_1, y_2 \) and \( y_3 \) by these relations in Eq. (28) leads to a differential system:

\[ \frac{dx}{dt}(t) = h(x(t), p) \]

where \( h \) is defined as:

\[ h(x, p) = f(x_1, x_2, y_1(x_1, x_2), y_2(x_1, x_2), y_3(x_1, x_2), p). \]

\((x_0(p), p)\) defines the steady flow of Eq. (40). Its expression depends on the parameters \( p \). For simplicity, we will consider an equivalent system (via a change of variable):

\[ \frac{dx}{dt}(t) = k(x(t), p) \]

where \( k \) satisfies \( k(0, p) = h(x_0(p), p) = 0 \).

Locally around the stability boundary Eq. (39), the period and the maximal amplitude of an unstable flow may be derived from a standard bifurcation analysis. Several methods may be used. For simplicity, we apply the *Normal Form Theorem* to our problem (Guckenheimer and Holmes, 1983). The system Eq. (41) is written in a simplified form by introducing a series of change of variable.

First, the system Eq. (41) is linearized around the equilibrium point \((x_0(p), p)\). Its eigenvalues are given by Eq. (38). When the parameters satisfy Eq. (39), let us say \( p = p_c \), these eigenvalues are pure and conjugate:

\[ \lambda_+(p_c) = i \omega, \quad \lambda_-(p_c) = -i \omega, \]

\[ \omega = \frac{1}{2} \left\{ \left( \text{Tr} \left[ P(p_c) - Q(p_c) [S(p_c)]^{-1} R(p_c) \right] \right)^2 \right. \]
\[ \left. - 4 \text{Det} \left[ P(p_c) - Q(p_c) [S(p_c)]^{-1} R(p_c) \right] \right\}^{1/2}. \]

When \( p = p_c \), the three order normal form of the linearized part of Eq. (41) is (Guckenheimer and Holmes, 1983):

\[ \frac{du}{dt}(t) = -\omega u + (au - bv)(u^2 + v^2), \]
\[ \frac{dv}{dt}(t) = \omega u + (av + bu)(u^2 + v^2) \]

where \( a \) and \( b \) depend on the derivatives of \( k \), at the point \((0, p_c)\). The dynamical behavior of this system is much easier to analyze and very close to the one coming from Eq. (41).
If we slightly modify the value of one parameter \( \mu \) around its critical value \( \mu_c \), a bifurcation occurs: the steady flow \((x_0(p), p)\) switches from a stable state to unstable state (or vice-versa). More precisely, if the following quantity

\[
d = \frac{1}{2} \left[ \frac{d}{d\mu} \text{Tr} \left[ P(\mu) - Q(\mu)[S(\mu)]^{-1} R(\mu) \right] \right]_{\mu=\mu_c} \tag{46}
\]
is nonzero, a Hopf bifurcation occurs: for each value of \( \mu \) close to \( \mu_c \) such that the steady flow \((x_0(p), p)\) is unstable, a periodic solution exists. It is a stable limit cycle if the coefficient \( a \) is strictly negative. Thus, every transient solutions close to \((x_0(p), p)\) converge to a limit cycle whether \( \mu \) is close to \( \mu_c \). One says that the bifurcation is supercritical.

In the next section, we will show on few examples that instabilities occurring in pipe-riser systems come from a supercritical Hopf bifurcation. Thus every slightly unstable flow converges to a limit cycle whose period is close to \( 2\pi/\omega \) (see Eq. (43)).

**EXPERIMENTAL VALIDATION**

Several years ago, Taitel Y. proposed a stability criterion for gas-liquid flows in pipe-riser systems (Taitel, 1986). This criterion was compared to experimental measurements at the University of Tulsa on a pipe-riser system connected to an additional pipe simulated by two volume variable tanks (cf. Fig. 6, page 9) (Taitel et al., 1990). \( l_p \) will refer to the length of this additional pipe. The Taitel criterion was satisfactory as long as \( l_p \) was greater than \( L \). Otherwise, instability prediction was all the more bad since \( l_p \) was small and improbable whenever \( l_p/L < 0.18 \). The practical interest of this criterion is therefore very limited.

In order to compare our stability criterion with the Tulsa experiments, we must take the additional pipe length \( l_p \) into account in our model. For that, we only replace the pressure equation Eq. (7) by the following equation:

\[
\frac{dP_{\text{pipe}}}{dt}(t) = \frac{U_G^0 - U_G^1(t)}{l_p + LR_{G}^{\text{pipe}}} P_{\text{pipe}}(t). \tag{47}
\]

Note that \( l_p \) is zero in a classical pipe-riser system (cf. Fig. 5, page 9). Therefore, \( l_p \) may be considered as an additional parameter.

Consequently, Eq. (22) is replaced by:

\[
F_3(X, Y, p) = \frac{U_G^0 - Y_1}{l_p + LR_{G}^{\text{pipe}}} X_3 \tag{48}
\]

and the fraction \((U_G^0 - Y_1)/(l_p + LR_{G}^{\text{pipe}})\) replaces \((U_G^0 - Y_1)/(LR_{G}^{\text{pipe}})\) in Eq. (27).

The following experiments are fully described in a Tulsa report (Vierkandt, 1988). In this paper, we will consider three flow maps plotted versus the parameters \( U_L^0 \) and \( U_G^0 \). Every parameters are constant, except the pipe inclination \( \theta_p \) respectively equal to \(-1^\circ, -2^\circ, \) and \(-5^\circ\) in figures 7, 8 and 9. The remaining parameter values are:

\[
P_S = 10^5 \text{ Pa, } L = 9.1 \text{ m, } l_p = 1.69 \text{ m, } H = 3 \text{ m, } D = 0.0254 \text{ m, } \rho_L = 998.2071 \text{ kg.m}^{-3}, \quad T = 293.15 \text{ K, } \sigma = 72.75 \times 10^{-3} \text{ N.m}^{-1}. \]

The Boe criterion is commonly used to predict severe slug flows (Boe, 1981):

\[
U_L^0 \geq \frac{P_{\text{atm}}}{\rho_L g(l_p + LR_{G}^{\text{pipe}})} U_G^0 \implies \text{severe slugging.} \tag{49}
\]

The Boe criterion line we plotted in this paper is a combination of this criterion and the following condition (see Eq. (3)):

\[
U_L^0 > 149 \left[ \frac{D}{4} \right]^{2/3} \left| \sin \theta_p \right|^{1/2} \implies \text{no severe slugging.} \tag{50}
\]

Since the flow pattern in the riser may be bubbly or slug, different values for \( C_0 \) and \( u_d \) may be chosen (see Eq. 9 and 10). Therefore, two stability criteria are plotted, one for slug flows and another one for bubble flows.

The points referred as "transition to severe slugging" represent flows with an unstable liquid column in the riser. Such flows generate liquid slugs at the riser outlet. Their length is less than the riser height. In severe slug flows, oscillations (in pressure, void fraction, etc.) are much higher and slug lengths always exceed the riser height.

We notice that the intersection between the regions bounded by the Boe criterion and the stability line (slug flow) gives a rather good prediction of unstable flows. Notice that the stability criterion with a bubble flow in the riser overestimates the unstable region.

The next experiments (cf. Fig. 10, page 10) are provided by Fabre, J. et al. (Fabre et al., 1990). The pipe inclination is \( \theta_p = -1\% \approx 0.57^\circ \) while the other parameter values are:

\[
P_S = 10^5 \text{ Pa, } L = 25 \text{ m, } l_p = 0 \text{ m, } H = 13.5 \text{ m, } \quad D = 0.053 \text{ m, } \rho_L = 998.2071 \text{ kg.m}^{-3}, \quad T = 293.15 \text{ K, } \quad \sigma = 72.75 \times 10^{-3} \text{ N.m}^{-1}. \]
Our stability criterion gives a good prediction of the unstable region when a bubble flow is assumed in the riser. Under a slug flow assumption, some flows are not predicted unstable ($U_G^0 = 1 \text{ m.s}^{-1}$ and $U_L^0 = 0.064$ or $0.127 \text{ m.s}^{-1}$). But, Fabre, J. et al. notice that it is often difficult to distinguish an unstable flow from a slug flow when liquid slugs do not have a significant length. These two examples may be close to that situation.

The last experimental flow map is provided by Schmidt, Z. et al. and is available in several papers (Schmidt et al., 1980), (Taitel, 1986). The variable parameters are still $U_G$ and $U_L$. The other ones are:

$$P_S = 10^5 \text{Pa}, \ L = 30.48 \text{ m}, \ l_p = 0 \text{ m}, \ H = 15.24 \text{ m},$$
$$D = 0.0508 \text{ m}, \ \theta_p = -5^\circ, \ \theta_r = 90^\circ, \ \rho_l = 824.95275 \text{ kg.m}^{-3},$$
$$T = 293.15 \text{ K}, \ \sigma = 28 \times 10^{-3} \text{ N.m}^{-1}.$$

Both regions bounded by our stability criterion with slug or bubble flow in the riser give a good prediction of the unstable region (cf. Fig. 11, page 10).

The last two figures validate our nonlinear analysis on two examples. Let us consider the experiments of Fabre, J. et al. We notice on the figure 10 that instability occurs whenever $U_G^0$ is decreased while $U_L^0$ is kept constant and small. We will choose $U_G^0$ as a bifurcation parameter. Given the same parameter values (see above) and $U_L^0 = 0.127 \text{ m.s}^{-1}$, a symbolic computation with Maple gives the bifurcation value of $U_G^0$, the pulsation $\omega$ (Eq. (43)) and the coefficients $a$ and $d$ (Eq. 44 - 46):

$$(U_G^0)_{\text{crit}} = 0.8293123557 \text{ m.s}^{-1}, \ \omega = 1.245578683,$$
$$a = -0.1355165705, \ d = -1.61556600.$$

Since $d$ is nonzero and $a$ is strictly negative, a variation of $U_G^0$ below the critical value $(U_G^0)_{\text{crit}}$ generates a supercritical Hopf bifurcation. Flows become unstable and converge to a periodic solution (limit cycle) for each value of $U_G^0$.

A more complex modelling of two-phase flow in pipeline-riser systems was recently presented in (Zakarian and Tran, 1999). This model was validated on a large set of experiments. Given all the aforementioned parameters and $U_L^0 = 0.829 \text{ m.s}^{-1} < (U_G^0)_{\text{crit}}$, transient solutions were numerically computed. We plotted the projections of these solutions on the plane $(m_L, m_G)$ (cf. Fig. 12, page 10). In addition to theses curves, we plotted the limit cycle predicted by the nonlinear analysis presented in this paper. We observe a very precise consistency between the limit cycle given by the numerical computations and the predicted periodic solution.

The same kind of comparison was done on the aforementioned experiments of Schmidt, Z. et al. (cf. Fig. 13, page 10). Again, numerical computations and symbolical analysis provided a similar prediction of the limit cycle for a given $U_G^0$.

CONCLUSION

A differential-algebraic model was presented to reproduce slightly unstable gas-liquid flows in pipe-riser systems. A linear analysis led to an analytical expression of the boundary between stable and unstable flows. This boundary is given as a function of physical parameters (pipe length, riser height, inlet phase flow rates, separator pressure, etc.) and provides a fairly accurate stability criterion.

Then, a nonlinear analysis gave the bifurcation curves of gas-liquid flows in pipe-riser systems, locally around their stability boundary. This analysis proved on few examples that the severe slugging phenomenon comes from a real hydrodynamic instability generated by a supercritical Hopf bifurcation.

Presently, our purpose is to extend this analysis to more complex systems such as production pipelines which are usually made up of a large number of pipe section inclined at different angles. The main purpose of this work is to find a prediction criterion for terrain instabilities such as the terrain slugging phenomenon (Linga, 1987).

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REFERENCES


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**Figure 1.** Steady flow with continuous gas penetration.

**Figure 2.** Flow patterns in horizontal pipes.

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Figure 3. Flow patterns in vertical pipes.

Figure 4. Wetted angle of a stratified flow.

Figure 5. Notations.

Figure 6. Tulsa experimental facility.

Figure 7. Tulsa experiments: $\theta_p = -1^\circ$.

Figure 8. Tulsa experiments: $\theta_p = -2^\circ$. 

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Figure 9. *Tulsa experiments*: $\theta_p = -5^\circ$.

Figure 10. *Fabre experiments*.

Figure 11. *Schmidt experiments*.

Figure 12. *Limit cycle prediction*.

Figure 13. *Limit cycle prediction*.